QUANTUM KAC'S CHAOS

GEORGE ANDROULAKIS AND RADE MUSULIN

Abstract. We study the notion of quantum Kac's chaos which was implicitly introduced by Spohn and explicitly formulated by Gottlieb. We prove the analogue of a result of Sznitman which gives the equivalence of Kac's chaos to 2-chaoticity and to convergence of empirical measures. Finally we give a simple, different proof of a result of Spohn which states that chaos propagates with respect to certain Hamiltonians that define the evolution of the mean field limit for interacting quantum systems.

1. The Motivation Behind Kac's Chaos

The origins of chaos, as discussed in this paper, dates back to Kac. In 1956, Kac [12] was interested in solving the non-linear integro-differential equation known as the Boltzmann equation [12, Equation (1.1)]. The solution to the Boltzmann equation is a family $(f^{(N)})_{N=1}^{\infty}$ of probability density functions, where $f^{(N)}$ describes the velocities and positions of N dilute gas molecules moving in $\mathbb{R}^3$, interacting via elastic binary collisions. The non-linearity of the Boltzmann equation provided difficulty in obtaining the existence of its solution.

If the gas is restricted to a container of fixed volume, there are no external forces, and the number N of molecules is assumed to be equidistributed, then $f^{(N)}$ depends on the velocities of the N gas molecules and time, thus having $3N+1$ real variables. Then the Boltzmann equation takes a simplified reduced form [12, Equation (1.3)] which is still a non-linear integro-differential equation. Further assuming that the kinetic energy of the system remains constant proportional to N, the $3N$ variables representing velocity lie on a sphere of radius $\sqrt{N}$ in $\mathbb{R}^{3N}$, and in order to obtain a further simplified version of the Bolzmann equation, one can replace the $3N$ real variables by one real variable x. This further reduces the Boltzmann equation to the reduced Boltzmann equation [12, Equation (3.5)]:

$$(1)\quad \frac{\partial f(x,t)}{\partial t} = \frac{\nu}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{2\pi}\{f(x\cos\theta+y\sin\theta,t)f(-x\sin\theta+y\cos\theta,t)-f(x,t)f(y,t)\}d\theta dy.$$

1991 *Mathematics Subject Classification.* Primary: 81Q50, Secondary: 35Q83, 37K99.

Key words and phrases. Kac's chaos, quantum Kac's chaos, empirical measure, Hartree equation.

The article is part of the second author's Ph.D. thesis which is prepared at the University of South Carolina under the supervision of the first author.

Kac further introduced a linear differential equation which he called the "Master Equation" [12, Equation (2.6)]. If $\phi^{(N)}$ is a solution to Kac's "Master Equation", $\phi_1^{(N)}$ and $\phi_2^{(N)}$ will denote the first and second marginals of $\phi^{(N)}$, respectively, i.e.

$$\phi_1^{(N)}(x,t) = \int_{x_2^2+\ldots+x_N^2=N-x^2} \phi^{(N)}(x,x_2,\ldots,x_N,t)d\sigma_1(x_2,\ldots,x_N)$$

and

$$\phi_2^{(N)}(x,y,t) = \int_{x_3^2+\ldots+x_N^2=N-x^2-y^2} \phi^{(N)}(x,y,x_3,\ldots,x_N,t)d\sigma_2(x_2,\ldots,x_N)$$

where σ_1, σ_2 are normalized uniform measures on the spheres of $\mathbb{R}^{N-1}$ and $\mathbb{R}^{N-2}$ respectively, centered at the origin and having radii $\sqrt{N-x^2}$ and $\sqrt{N-x^2-y^2}$ respectively. Kac [12] noticed that if the pointwise limits $\lim_{N\to\infty} \phi_1^{(N)}(x,0)$ and $\lim_{N\to\infty} \phi_2^{(N)}(x,0)$ exist for all $x \in \mathbb{R}$, and

$$\lim_{N\to\infty} \phi_2^{(N)}(x,y,0) = \lim_{N\to\infty} \phi_1^{(N)}(x,0) \lim_{N\to\infty} \phi_1^{(N)}(y,0), \tag{2}$$

then the same limits exist at any later time t, and satisfy

$$\lim_{N\to\infty} \phi_2^{(N)}(x,y,t) = \lim_{N\to\infty} \phi_1^{(N)}(x,t) \lim_{N\to\infty} \phi_1^{(N)}(y,t). \tag{3}$$

Then equation (3) implies that the function f defined by

$$f(x,t) := \lim_{N\to\infty} \phi_1^{(N)}(x,t)$$

satisfies equation (1). Hence Kac proved the existence of the solution to the reduced Boltzmann equation for $N=1$. Kac [12] referred to the property in equation (3) for a fixed $t \geq 0$ as the "Boltzmann property". Whenever equation (2) implies equation (3) for all times $t > 0$, we say that the "Boltzmann property propagates in time". Hence Kac [12] proved that the Boltzmann property propagates in time for his "Master Equation".

Many authors including McKean [13], Johnson [11], Tanaka [19], Ueno [20], Grünbaum [10], Graham and Méléard [9], Sznitman [18], Mischler [14], Carlen, Carvalho and Loss [5], Mischler and Mouhot [15] have abstracted the idea of the "Boltzmann property" to a sequence of probability measures on a topological space. Instead of having the "Boltzmann property", the sequence of probability measures nowadays are said to be chaotic. In order to discuss chaotic sequences of probability measures, these authors first define the notion of a symmetric probability measure.

Definition 1.1. *Let E be a topological space, N be a positive integer, μ_N be a probability measure on the Borel subsets of E^N. Then μ_N is called* ***symmetric*** *if for any N-many* *continuous real-valued bounded functions on E, $\phi_1, \phi_2, ..., \phi_N$,*

$$\int_{E^N} \phi_1(x_1)\phi_2(x_2)\cdots\phi_N(x_N)d\mu_N = \int_{E^N} \phi_1(x_{\pi(1)})\phi_2(x_{\pi(2)})\cdots\phi_n(x_{\pi(N)})d\mu_N$$

for any permutation π of $\{1,\ldots,N\}$.

A chaotic sequence of probability measures is then defined as follows.

Definition 1.2. *Let E be a topological space, μ be a Borel probability measure on E, and for every $N \in \mathbb{N}$ let μ_N be a symmetric Borel probability measure on E^N. For $k \in \mathbb{N}$, we say that $(\mu_N)_{N=1}^{\infty}$ is $k-\mu$-**chaotic** if for every choice $\phi_1, \phi_2, ..., \phi_k$ of continuous bounded real-valued functions on E, we have*

$$\lim_{N\to\infty} \int_{E^N} \phi_1(x_1)\phi_2(x_2)\cdots\phi_k(x_k)d\mu_N = \prod_{j=1}^{k}\int_E \phi_j(x)d\mu(x).$$

*We say that $(\mu_N)_{N=1}^{\infty}$ is μ-**chaotic** if $(\mu_N)_{N=1}^{\infty}$ is $k-\mu$-chaotic for all $k \geq 1$.*

Boltzmann's equation and Equation (1) describe evolutions in models of classical mechanics. Corresponding quantum mechanical models are described in [16, V. Quantum Mechanical Models]. In such models, density functions are replaced by **density operators**, (positive operators of trace equal to 1), which via the trace duality define states on algebras of bounded linear operators acting on Hilbert spaces. The corresponding notion to the chaotic sequences of probability measures, as well as the corresponding notion to the propagation of chaos appears in [16, Theorem 5.7] where the time evolution is given by a specific family of Hamiltonians. Gottlieb [8] formulated the notion of chaotic sequences of density operators. In the current article, we study the notion of chaos which was introduced by Spohn and formalized by Gottlieb. To honor the fact that the definition of chaos was originated by the work of Kac for classical models, we refer to its quantum version as "quantum Kac's chaos". We prove two main results in this article. The first result is our Theorem 2.11 which is the analogue of [18, Proposition 2.2(i)]. The second result of this article is our Theorem 3.5 which is a simpler, different proof of the propagation of chaos result of Spohn [16, Theorem 5.7]. This result shows that chaos propagates in the mean field limit for interacting quantum systems.

Notation: Throughout this paper, $\mathbb{H}$ will denote an arbitrary Hilbert space, $\mathcal{B}(\mathbb{H})$ will denote the set of bounded operators on $\mathbb{H}$, and $\mathcal{D}(\mathbb{H})$ will denote the set of density operators on $\mathbb{H}$. The identity operator on $\mathcal{B}(\mathbb{H})$ will be denoted by 1. For any operator $A \in \mathcal{B}(\mathbb{H})$ and $k \in \mathbb{N}$, $A^{\otimes k}$ will denote the tensor product of A with itself k times. In addition, for any $A \in \mathcal{B}(\mathbb{H})$, $||A||_\infty$ will denote the $\mathcal{B}(\mathbb{H})$ norm of A. If A is a trace class operator on $\mathbb{H}$, then $||A||_1$ will denote the trace class norm of A.

For any $k, N \in \mathbb{N}$ with $k \leq N$, and $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$, we will denote by $\rho_N^{(k)} \in \mathcal{D}(\mathbb{H}^{\otimes k})$ the partial trace of ρ_N where we trace out all but the first k copies of $\mathbb{H}$. In addition, for an index set $A \subset \{1, ..., N\}$, we will denote by $\text{tr}_A(\rho_N)$ the partial trace of ρ_N where we trace out the copies of $\mathbb{H}$ indexed by elements of A. Notice that $\text{tr}_{[k+1,N]}(\rho_N) = \rho_N^{(k)}$.

Given a separable metric space E, we will denote by $M(E)$ the set of probability measures on E. The set of continuous bounded real-valued functions on E will be denoted by $C_b(E)$. Finally, for $N \in \mathbb{N}$, Σ_N will denote the set of all permutations of the set $\{1, 2, ..., N\}$.

Acknowledgement: The authors would like to thank Eric Carlen for bringing the topic of quantum Kac's chaos to our attention.

2. Equivalent Statements of Quantum Kac's Chaos

Sznitman used probabilistic methods to show existence [17] and uniqueness [18] to the homogeneous Boltzmann equation. The next result was important in his proofs.

Proposition 2.1. [18, Proposition 2.2] *Let E be a separable metric space, $(\mu_N)_{N=1}^{\infty}$ a sequence of symmetric probability measures on E^N, and μ be a probability measure on E. The following are equivalent:*

1. *The sequence $(\mu_N)_{N=1}^{\infty}$ is μ-chaotic (as in Definition 1.2).*
2. *The function $X_N : E^N \to M(E)$ defined by $X_N(x_1, ..., x_N) = \dfrac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$ (where δ_x stands for the Dirac measure at x), converges in law with respect to μ_N to the constant random variable μ, i.e. for every $g \in C_b(E)$ we have that*
$$\int_{E^N} |(X_N - \mu)g|^2 d\mu_N \xrightarrow[N\to\infty]{} 0.$$
3. *The sequence $(\mu_N)_{N=1}^{\infty}$ is $2-\mu$-chaotic (as in Definition 1.2).*

The main result of this section is to obtain a quantum analogue of Proposition 2.1. Instead of considering probability density functions, we consider density operators. We first have to extend the definition of symmetric measures (Definition 1.1) to density operators. The following is the quantum version of symmetry (Definition 1.1) we will use in this paper.

Definition 2.2. *Let $N \in \mathbb{N}$. A density operator $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$ is **symmetric** if and only if for every $A_1, ..., A_N \in \mathcal{B}(\mathbb{H})$ and for every permutation $\pi \in \Sigma_N$,*
$$tr(A_1 \otimes \cdots \otimes A_N \rho_N) = tr(A_{\pi(1)} \otimes \cdots \otimes A_{\pi(N)} \rho_N).$$

This is not the same formulation of the definition of symmetric density operators given by Gottlieb [8]. To obtain the formulation given by Gottlieb [8], for $N \in \mathbb{N}$, define for each

$\pi \in \Sigma_N$ the unitary operator $U_\pi^{[N]} \in \mathcal{B}(H^{\otimes N})$ by

$$U_\pi^{[N]}(x_1 \otimes \cdots \otimes x_N) = x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(N)}. \tag{4}$$

A density operator $\rho_N \in \mathcal{B}(\mathbb{H}^{\otimes N})$ is symmetric according to [8] if and only if $U_\pi^{[N]}\rho_N = \rho_N U_\pi^{[N]}$ for every $\pi \in \Sigma_N$. However, Gottlieb's definition of symmetric densities is equivalent to Definition 2.2 as we show next.

Proposition 2.3. *Let $N \in \mathbb{N}$ and $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$. Then ρ_N is symmetric (as in Definition 2.2) if and only if $U_\pi^{[N]}\rho_N = \rho_N U_\pi^{[N]}$ for all $\pi \in \Sigma_N$.*

Proof. ($\Rightarrow$) Let $\pi \in \Sigma_N$. Then

$$\begin{aligned} \operatorname{tr}(A_1 \otimes \cdots \otimes A_N \rho_N) &= \operatorname{tr}(A_{\pi(1)} \otimes \cdots \otimes A_{\pi(N)}\rho_N) = \operatorname{tr}(U_{\pi^{-1}}^{[N]}(A_1 \otimes \cdots \otimes A_N)U_\pi^{[N]}\rho_N) \\ &= \operatorname{tr}((A_1 \otimes \cdots \otimes A_N)U_\pi^{[N]}\rho_N U_{\pi^{-1}}^{[N]}) \end{aligned}$$

for any $A_1, ..., A_N \in \mathcal{B}(\mathbb{H}^{\otimes N})$. Since the set of arbitrary sums of simple tensors of bounded operators in $\mathcal{B}(\mathbb{H})$ is dense in $\mathcal{B}(\mathbb{H}^{\otimes N})$, this is equivalent to $U_\pi^{[N]}\rho_N U_{\pi^{-1}}^{[N]} = \rho_N$, i.e. $U_\pi^{[N]}\rho_N = \rho_N U_\pi^{[N]}$.

($\Leftarrow$) For each $\pi \in \Sigma_N$,

$$\begin{aligned} \operatorname{tr}(A_1 \otimes \cdots \otimes A_N \rho_N) &= \operatorname{tr}(A_1 \otimes \cdots \otimes A_N U_\pi^{[N]}\rho_N U_{\pi^{-1}}^{[N]}) = \operatorname{tr}(U_{\pi^{-1}}^{[N]}(A_1 \otimes \cdots \otimes A_N)U_\pi^{[N]}\rho_N) \\ &= \operatorname{tr}(A_{\pi(1)} \otimes \cdots \otimes A_{\pi(N)}\rho_N). \end{aligned}$$

$\square$

Some examples of symmetric density operators are as follows.

Example 2.4. *Let $\rho \in \mathcal{D}(\mathbb{H})$. For any $N \in \mathbb{N}$, define $\rho_N := \rho^{\otimes N}$. It is clear that ρ_N is symmetric.*

Example 2.5. *Let $N \in \mathbb{N}$ and $B_1, ..., B_N \in \mathcal{D}(\mathbb{H})$. Then*

$$\rho_N := \frac{1}{N!}\sum_{\sigma \in \Sigma_N} B_{\sigma(1)} \otimes \cdots \otimes B_{\sigma(N)} \in \mathcal{D}(\mathbb{H}^{\otimes N})$$

is symmetric.

Example 2.6. *Let $(E, \mathcal{F}, P)$ be a probability space, $N \in \mathbb{N}$, and $\mu_N \in M(E^N)$ be symmetric. Then for any measurable, bounded, and integrable, (in the Bochner integral sense), function $D : E \to \mathcal{D}(\mathbb{H})$, the density operator $D_N \in \mathcal{D}(\mathbb{H})$ defined by*

$$D_N := \int_{E^N} D(\omega_1) \otimes D(\omega_2) \otimes \cdots \otimes D(\omega_N) d\mu_N(\omega_1, \omega_2, ..., \omega_N)$$

is symmetric.

Proof. For each $\pi \in \Sigma_N$ and $A_1, ..., A_N \in \mathcal{B}(\mathbb{H})$,

$$\begin{aligned}
\mathrm{tr}(A_1 \otimes \cdots \otimes A_N D_N) &= \mathrm{tr}(A_1 \otimes \cdots \otimes A_N \int_{E^N} D(\omega_1) \otimes D(\omega_2) \otimes \cdots \otimes D(\omega_N) d\mu_N) \\
&= \int_{E^N} \mathrm{tr}(A_1 D(\omega_1))\mathrm{tr}(A_2 D(\omega_2)) \cdots \mathrm{tr}(A_N D(\omega_N)) d\mu_N \\
&= \int_{E^N} \mathrm{tr}(A_1 D(\omega_{\pi^{-1}(1)})\mathrm{tr}(A_2 D(\omega_{\pi^{-1}(2)})) \cdots \mathrm{tr}(A_N D(\omega_{\pi^{-1}(N)})) d\mu_N \\
&= \int_{E^N} \mathrm{tr}(A_{\pi(1)} D(\omega_1))\mathrm{tr}(A_{\pi(2)} D(\omega_2)) \cdots \mathrm{tr}(A_{\pi(N)} D(\omega_N)) d\mu_N \\
&= \mathrm{tr}(A_{\pi(1)} \otimes \cdots \otimes A_{\pi(N)} \int_{E^N} D(\omega_1) \otimes D(\omega_2) \otimes \cdots \otimes D(\omega_N) d\mu_N) \\
&= \mathrm{tr}(A_{\pi(1)} \otimes \cdots \otimes A_{\pi(N)} D_N).
\end{aligned}$$

□

The following is the quantum version of Definition 1.2 that we will use in this paper.

Definition 2.7. *Let $(\rho_N)_{N=1}^{\infty}$ be a sequence of symmetric density operators such that $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$ for each $N \in \mathbb{N}$, $\rho \in \mathcal{D}(\mathbb{H})$ be a density operator, and $k \in \mathbb{N}$. Then $(\rho_N)_{N=1}^{\infty}$ is $k - \rho$-**chaotic** if and only if for all $A_1, ..., A_k \in \mathcal{B}(\mathbb{H})$,*

$$tr(A_1 \otimes \cdots \otimes A_k \otimes 1^{\otimes(N-k)} \rho_N) \xrightarrow[N\to\infty]{} \prod_{j=1}^{k} tr(\rho A_j).$$

*We say that $(\rho_N)_{N=1}^{\infty}$ is ρ-**chaotic** if and only if $(\rho_N)_{N=1}^{\infty}$ is $k - \rho$-chaotic for all $k \geq 1$.*

Next we give many equivalent formulations of this definition. We will use the fact that the partial trace of a density operator is a density operator.

Proposition 2.8. *Let $(\rho_N)_{N=1}^{\infty}$ be a sequence of symmetric density matrices such that $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$ for each $N \in \mathbb{N}$, $\rho \in \mathcal{D}(\mathbb{H})$, and $k \in \mathbb{N}$. The following are equivalent*

1. *$(\rho_N)_{N=1}^{\infty}$ is $k - \rho$-chaotic,*
2. *$tr\left(\left(\rho_N^{(k)} - \rho^{\otimes k}\right) A_1 \otimes \cdots \otimes A_k\right) \xrightarrow[N\to\infty]{} 0$ for all $A_1, ..., A_k \in \mathcal{B}(\mathbb{H})$,*
3. *$tr\left(\left(\rho_N^{(k)} - \rho^{\otimes k}\right) A\right) \xrightarrow[N\to\infty]{} 0$ for all $A \in \mathcal{B}(\mathbb{H}^{\otimes k})$, and*
4. *$tr|\rho_N^{(k)} - \rho^{\otimes k}| \xrightarrow[N\to\infty]{} 0.$*

Proof. ((1) ⇔ (2)) This is obvious. See Attal [1, Theorem 2.28].
((2) ⇔ (3)) This follows from the fact that the set of sums of simple tensors of bounded operators in $\mathcal{B}(\mathbb{H})$ is dense in $\mathcal{B}(\mathbb{H}^{\otimes N})$.

((3) $\Leftrightarrow$ (4)) Wehrl [21, Theorem 3] proved that a sequence, $(D_N)_{N=1}^{\infty} \subset \mathcal{D}(\mathbb{K})$, of density operators on a Hilbert space $\mathbb{K}$ converges weakly to a density operator $D \in \mathcal{D}(\mathbb{K})$ if and only if it converges in norm, i.e. $\operatorname{tr}|D_N - D| \xrightarrow[N\to\infty]{} 0$. Our result follows by letting $\mathbb{K} = \mathbb{H}^{\otimes k}$, $D_N = \rho_N^{(k)}$ for each N, and $D = \rho^{\otimes k}$. □

Condition (4) in Proposition 2.8 appears in [16, Theorem 5.7]. Gottlieb [8] used this condition for all k to define that "ρ_N is ρ-chaotic". Proposition 2.8 shows that Gottlieb's definition agrees with ours. We will now give some examples of chaotic sequences.

Example 2.9. *Let $\rho \in \mathcal{B}(\mathbb{H})$. For each $N \in \mathbb{N}$, define $\rho_N := \rho^{\otimes N}$. Then it is clear that $(\rho_N)_{N=1}^{\infty}$ is ρ-chaotic.*

The following example due to Gottlieb [8, Lemma 1.3.2] gives a way of constructing a chaotic sequence of density operators from any classically chaotic sequence of probability measures.

Example 2.10. *Let $(E, \mathcal{F}, P)$ be a probability space and $(\mu_N)_{N=1}^{\infty}$ be a sequence of symmetric probability measures which are μ-chaotic for some probability measure $\mu \in M(E)$. Let $D : E \to \mathcal{D}(\mathbb{H})$ be a measurable, bounded, and integrable function, (in the Bochner integral sense). Define $D_N := \int_{E^N} D(\omega_1) \otimes D(\omega_2) \otimes \cdots \otimes D(\omega_N) d\mu_N(\omega_1, \omega_2, ..., \omega_N)$ and $\overline{D} := \int_E D(\omega) d\mu$. We know from Example 2.6 that $D_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$ is a symmetric density operator for each $N \in \mathbb{N}$. Then $(D_N)_{N=1}^{\infty}$ is $\overline{D}$-chaotic.*

Proof. For any $k \geq 1$ and $A_1, ..., A_k \in \mathcal{B}(\mathbb{H})$,

$$\begin{aligned} & \operatorname{tr}(A_1 \otimes \cdots \otimes A_k \otimes 1^{\otimes(N-k)} \int D(\omega_1) \otimes \cdots \otimes D(\omega_N) d\mu_N) \\ = \ & \operatorname{tr}\left(\int A_1 D(\omega_1) \otimes \cdots \otimes A_k D(\omega_k) \otimes D(\omega_{k+1}) \otimes \cdots \otimes D(\omega_N) d\mu_N \right) \\ = \ & \int \operatorname{tr}(A_1 D(\omega_1)) \cdots \operatorname{tr}(A_k D(\omega_k)) d\mu_N \end{aligned}$$

which converges to

$$\int \operatorname{tr}(A_1 D(\omega_1)) \cdots \operatorname{tr}(A_k D(\omega_k)) d\mu^{\otimes k} = \prod_{j=1}^{k} \operatorname{tr}\left(A_j \int D(\omega) d\mu \right)$$

as N approaches infinity by Definition 1.2. □

Now we are ready to prove the analogous statement to Proposition 2.1 ([18, Proposition 2.2]) for chaotic sequences of density operators.

Theorem 2.11. *Let $(\rho_N)_{N=1}^\infty$ be a symmetric sequence of density operators such that $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$ for each $N \in \mathbb{N}$, and let $\rho \in \mathcal{D}(\mathbb{H})$. Then the following are equivalent.*

(1) *$(\rho_N)_{N=1}^\infty$ is $k - \rho$-chaotic for all $k \in \mathbb{N}$,*
(2) *$(\rho_N)_{N=1}^\infty$ is $2 - \rho$-chaotic, and*
(3) *for each $A \in \mathcal{B}(\mathbb{H})$,*

$$tr\left(\left|\frac{1}{N}\sum_{j=1}^{N} 1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j)} - tr(A\rho)1^{\otimes N}\right|^2 \rho_N\right) \xrightarrow[N\to\infty]{} 0.$$

The function $A \to \frac{1}{N}\sum_{j=1}^{N} 1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j)}$ is studied in [7] and is called a quantum empirical measure. The above theorem and [18, Proposition 2.2] gives more justifications for the choice of this term.

Proof. ((1) $\Rightarrow$ (2)) This is obvious.

((2) $\Rightarrow$ (3)) Let $A \in \mathcal{B}(\mathbb{H})$. Notice that

$$\mathrm{tr}\left(\left|\frac{1}{N}\sum_{j=1}^{N} 1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j)} - \mathrm{tr}(A\rho)1^{\otimes N}\right|^2 \rho_N\right) \tag{5}$$

$$= \mathrm{tr}\Big(\Big(\frac{1}{N}\sum_{i=1}^{N} 1^{\otimes(i-1)}\otimes A^*\otimes 1^{\otimes(N-i)} - \overline{\mathrm{tr}(A\rho)}1^{\otimes N}\Big)\Big(\frac{1}{N}\sum_{j=1}^{N} 1^{\otimes(j-1)}\otimes A\otimes 1^{\otimes(N-j)} - \mathrm{tr}(A\rho)1^{\otimes N}\Big)\rho_N\Big).$$

By distributing, we obtain that the last expression is equal to

$$\frac{1}{N^2}\mathrm{tr}\Big(\sum_{i,j=1}^{N}(1^{\otimes(i-1)} \otimes A^* \otimes 1^{\otimes(N-i)})(1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j)})\rho_N\Big) \tag{6}$$

$$-\frac{\mathrm{tr}(A\rho)}{N}\mathrm{tr}\Big(\sum_{j=1}^{N} 1^{\otimes(j-1)} \otimes A^* \otimes 1^{\otimes(N-j)}\rho_N\Big) \tag{7}$$

$$-\frac{\overline{\mathrm{tr}(A\rho)}}{N}\mathrm{tr}\Big(\sum_{j=1}^{N} 1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j)}\rho_N\Big) \tag{8}$$

$$+ |\mathrm{tr}(A\rho)|^2. \tag{9}$$

We will obtain that the sum of lines (6), (7), (8), and (9) goes to zero as N approaches infinity. To evaluate (6), we consider three cases: when $i = j$, when $i < j$, and when $j < i$.

If $i = j$, then by symmetry of ρ_N,

$$\begin{aligned}\frac{1}{N^2}\sum_{j=1}^{N} \mathrm{tr}(1^{\otimes(j-1)} \otimes |A|^2 \otimes 1^{\otimes(N-j)}\rho_N) &= \frac{1}{N}\mathrm{tr}(|A|^2 \otimes 1^{\otimes(N-1)}\rho_N) \\ &\leq \frac{1}{N}|||A|^2 \otimes 1^{\otimes(N-1)}||_\infty ||\rho_N||_1 \leq \frac{|||A|^2||_\infty}{N},\end{aligned}$$

which goes to zero as N approaches infinity. If $i < j$, then by symmetry of ρ_N,

$$\begin{aligned}\frac{1}{N^2}\sum_{i<j} \mathrm{tr} \quad &(1^{\otimes(i-1)} \otimes A^* \otimes 1^{\otimes(j-i-1)} \otimes A \otimes 1^{\otimes(N-j)}\rho_N) = \frac{1}{N^2}\frac{N!}{2(N-2)!}\mathrm{tr}(A^* \otimes A \otimes 1^{\otimes(N-2)}\rho_N) \\ &= \frac{N-1}{2N}\mathrm{tr}(A^* \otimes A \otimes 1^{\otimes(N-2)}\rho_N) \xrightarrow[N\to\infty]{2-\rho-\text{chaotic}} \frac{1}{2}\mathrm{tr}(A\rho)\mathrm{tr}(A^*\rho) = \frac{1}{2}|\mathrm{tr}(A\rho)|^2.\end{aligned}$$

If $j < i$ we obtain exactly the same limit. Thus, we have that line (6) converges to $|\mathrm{tr}(A\rho)|^2$ as N approaches infinity.

Using symmetry of ρ_N and by assumption, we obtain the limit of line (7),

$$\begin{aligned}\frac{-\mathrm{tr}(A\rho)}{N} \quad \mathrm{tr} \quad &(\sum_{j=1}^{N} 1^{\otimes(j-1)} \otimes A^* \otimes 1^{\otimes(N-j)}\rho_N) \\ &= -\mathrm{tr}(A\rho)\mathrm{tr}(A^* \otimes 1^{\otimes(N-1)}\rho_N) \xrightarrow[N\to\infty]{} -\mathrm{tr}(A\rho)\mathrm{tr}(A^*\rho) = -|\mathrm{tr}(A\rho)|^2,\end{aligned}$$

where in the last limit, we used the obvious fact that if ρ_N is $2-\rho$-chaotic then it is $1-\rho$-chaotic. Similarly, line (8) converges to $-|\mathrm{tr}(A\rho)|^2$ as N approaches infinity.

Therefore, the sum of lines (6), (7), (8), and (9) converge to

$$|\mathrm{tr}(A\rho)|^2 - |\mathrm{tr}(A\rho)|^2 - |\mathrm{tr}(A\rho)|^2 + |\mathrm{tr}(A\rho)|^2 = 0,$$

and line (5) converges to 0 as N approaches infinity.

((3) $\Rightarrow$ (1)) Let $k \in \mathbb{N}$ and $A_1, ..., A_k \in \mathcal{B}(\mathbb{H})$. Then

$$\left|\operatorname{tr}\left(A_1 \otimes \cdots \otimes A_k \otimes 1^{\otimes(N-k)}\rho_N\right) - \prod_{j=1}^{k} \operatorname{tr}(\rho A_j)\right| \leq \tag{10}$$

$$\Big|\operatorname{tr}\left(A_1 \otimes \cdots \otimes A_k \otimes 1^{\otimes(N-k)}\rho_N\right) \tag{11}$$

$$- \quad \operatorname{tr}\left(\prod_{j=1}^{k} \frac{1}{N}\left(A_j \otimes 1^{\otimes(N-1)} + 1 \otimes A_j \otimes 1^{\otimes(N-2)} + \cdots + 1^{\otimes(N-1)} \otimes A_j\right)\rho_N\right)\Big|$$

$$+ \quad \Big|\operatorname{tr}\left(\prod_{j=1}^{k} \frac{1}{N}\left(A_j \otimes 1^{\otimes(N-1)} + 1 \otimes A_j \otimes 1^{\otimes(N-2)} + \cdots + 1^{\otimes(N-1)} \otimes A_j\right)\rho_N\right) \tag{12}$$

$$- \quad \prod_{j=1}^{N} \operatorname{tr}(\rho A_j)\Big|.$$

We label the first and second lines after the inequality by (11) and the third and fourth lines after the inequality by (12). Our goal will be to show that the sum of lines (11) and (12) goes to 0 as N approaches infinity.

For lines (11), for $k \leq N$ we define $E_{k,N}$ to be the set of embedings (i.e. one-to-one maps) $\sigma : \{1, ..., k\} \to \{1, ..., N\}$. Notice that $\# E_{k,N} = \dfrac{N!}{(N-k)!}$. Furthermore, for $\sigma \in E_{k,N}$ and $i \in \{1, ..., N\}$, define

$$A_{\sigma,i} := \begin{cases} A_j & \text{when } \sigma(j) = i \\ 1 & \text{otherwise.} \end{cases}$$

Then, by symmetry of ρ_N, we can rewrite lines (11) as

$$\Bigg|\operatorname{tr}\Bigg(\Bigg(\frac{(N-k)!}{N!} \sum_{\sigma \in E_{k,N}} \left[A_{\sigma,1} \otimes A_{\sigma,2} \otimes \cdots \otimes A_{\sigma,N}\right] \tag{13}$$

$$-\frac{1}{N^k} \prod_{j=1}^{k} \left(A_j \otimes 1^{\otimes(N-1)} + 1 \otimes A_j \otimes 1^{\otimes(N-2)} + \cdots + 1^{\otimes(N-1)} \otimes A_j\right)\Bigg)\rho_N\Bigg)\Bigg|. \tag{14}$$

In line (14), there are two types of terms: the terms with $N-k$ 1's in the expanded form which we call the off-diagonal terms, and all the other terms which we call the diagonal terms. There are $\dfrac{N!}{(N-k)!}$ off-diagonal terms and $N^k - \dfrac{N!}{(N-k)!}$ diagonal terms. Let $M := \max_{j=1,...,k} ||A_j||_\infty$. The off-diagonal terms are exactly the terms of line (13). Thus, the addition of line (13) and the off-diagonal terms of line (14) is bounded by

$$\left|\left|\left(\frac{(N-k)!}{N!}-\frac{1}{N^k}\right)\sum_{\sigma\in E_{k,N}} A_{\sigma,1}\otimes\cdots\otimes A_{\sigma,N}\right|\right|_\infty ||\rho_N||_1 \leq \frac{N!}{(N-k)!}\left(\frac{(N-k)!}{N!}-\frac{1}{N^k}\right)M^k.$$

Each diagonal term is also bounded by M^k. Thus, the diagonal terms of line (14) are bounded by $\frac{1}{N^k}\left(N^k-\frac{N!}{(N-k)!}\right)M^k$. Hence, we can bound lines (11) and take the limit as N approaches 0,

$$\begin{aligned}
&\frac{N!}{(N-k)!}\left(\frac{(N-k)!}{N!}-\frac{1}{N^k}\right)M^k+\frac{1}{N^k}\left(N^k-\frac{N!}{(N-k)!}\right)M^k\\
=\;& M^k\left[\left(\frac{N^k(N-k)!}{N!}-1\right)\frac{N!}{N^k(N-k)!}+\frac{N!}{N^k(N-k)!}\left(\frac{N^k(N-k)!}{N!}-1\right)\right]\\
=\;& 2M^k\left[\frac{N!}{N^k(N-k)!}\left(\frac{N^k(N-k)!}{N!}-1\right)\right]=2M^k\left[1-\frac{N!}{N^k(N-k)!}\right]\xrightarrow[N\to\infty]{}0.
\end{aligned}$$

So line (11) goes to 0 as N approaches infinity.

For lines (12), we define $\overline{X}_N : \mathcal{B}(\mathbb{H})\to\mathcal{B}(\mathbb{H}^{\otimes N})$ by

$$\overline{X}_N(A) := \frac{1}{N}\sum_{j=1}^{N} 1^{\otimes(j-1)}\otimes A\otimes 1^{\otimes(N-j)}.$$

Then, lines (12) can be rewritten as

$$\begin{aligned}
&\left|\mathrm{tr}\left[\left(\prod_{j=1}^{k}\overline{X}_N(A_j)-\prod_{j=1}^{k}\mathrm{tr}(\rho A_j)1\right)\rho_N\right]\right|\\
=\;&\left|\sum_{l=0}^{k-1}\mathrm{tr}\left[\left(\prod_{j=1}^{l}\mathrm{tr}(\rho A_j)\prod_{j=l+1}^{k}\overline{X}_N(A_j)-\prod_{j=1}^{l+1}\mathrm{tr}(\rho A_j)\prod_{j=l+2}^{k}\overline{X}_N(A_j)1^{\otimes N}\right)\rho_N\right]\right|\\
=\;&\left|\sum_{l=0}^{k-1}\mathrm{tr}\left[\left(\overline{X}_N(A_{l+1})-\mathrm{tr}(\rho A_{l+1})1^{\otimes N}\right)\prod_{j=1}^{l}\mathrm{tr}(\rho A_j)\prod_{j=l+2}^{k}\overline{X}_N(A_j)\rho_N\right]\right|,
\end{aligned}$$

and can be bounded by

$$\sum_{l=0}^{k-1}\left|\mathrm{tr}\left[\left(\overline{X}_N(A_{l+1})-\mathrm{tr}(\rho A_{l+1})1^{\otimes N}\right)\prod_{j=1}^{l}\mathrm{tr}(\rho A_j)\prod_{j=l+2}^{k}\overline{X}_N(A_j)\rho_N\right]\right|$$

$$\leq \sum_{l=0}^{k-1}\left(\mathrm{tr}\left[\left|\overline{X}_N(A_{l+1})-\mathrm{tr}(\rho A_{l+1})1^{\otimes N}\right|^2\rho_N\right]^{1/2}\cdot\mathrm{tr}\left[\left|\prod_{j=1}^{l}\mathrm{tr}(\rho A_j)\prod_{j=l+2}^{k}\overline{X}_N(A_j)\right|^2\rho_N\right]^{1/2}\right).$$

By assumption, for each l,

$$\mathrm{tr}\left[\left|\overline{X}_N(A_{l+1})-\mathrm{tr}(\rho A_{l+1})1^{\otimes N}\right|^2\rho_N\right]^{1/2}\xrightarrow[N\to\infty]{}0,$$

and if $M:=\max_{j=1,\ldots,k}||A_j||_\infty$, since $||\overline{X}_N(A_j)||_\infty\leq||A_j||_\infty\leq M$, we have that

$$\mathrm{tr}\left[\left|\prod_{j=1}^{l}\mathrm{tr}(\rho A_j)\prod_{j=l+2}^{k}\overline{X}_N(A_j)\right|^2\rho_N\right]\leq\prod_{j=1}^{l}|\mathrm{tr}(\rho A_j)|^2\mathrm{tr}\left[\left|\prod_{j=l+2}^{k}\overline{X}_N(A_j)\right|^2\rho_N\right]$$

$$\leq\prod_{j=1}^{l}|\mathrm{tr}(\rho A_j)|^2\left|\left|\prod_{j=l+2}^{k}\overline{X}_N(A_j)\right|\right|_\infty^2||\rho_N||_1\leq M^{2k}\prod_{j=1}^{l}|\mathrm{tr}(\rho A_j)|^2$$

which is bounded independent of N. Hence, lines (12) converges to 0 as N goes to infinity. Therefore, line (10) converges to 0 as N approaches infinity.

□

3. Propagation of Chaos

Spohn proved that under evolutions governed by certain families of Hamiltonians, chaotic sequences of density operators propagate in time [16, Theorem 5.7]. In this section, we will use the ideas of the proofs of Ducomet [6, Theorem 3.1], and Bardos, Golse, Gottlieb, and Mauser [2, Theorem 3.1] to give a simple, different proof to the result of Spohn. First, we define propagation of chaos.

Definition 3.1. *Let $(\rho_N(0))_{N=1}^\infty$ be a sequence of density operators and let $(H_N)_{N=1}^\infty$ be a sequence of Hamiltonians where $\rho_N(0)\in\mathcal{D}(\mathbb{H}^{\otimes N})$ and $H_N\in\mathcal{B}(\mathbb{H}^{\otimes N})$ for every $N\in\mathbb{N}$. For each $t\geq 0$ and $N\in\mathbb{N}$, define the density operator*

$$\rho_N(t):=e^{-itH_N}\rho_N(0)e^{itH_N}\in\mathcal{D}(\mathbb{H}^{\otimes N}).\tag{15}$$

If, for each fixed $t\geq 0$, the sequence $(\rho_N(t))_{N=1}^\infty$ is $\rho(t)$-chaotic for some $\rho(t)\in\mathcal{D}(\mathbb{H})$, then we say that chaos ***propagates with respect to*** *$(H_N)_{N=1}^\infty$.*

We will now construct, as in Spohn [16], examples of propagation of chaos. We will examine the mean field limit for interacting quantum particles, see [16, pages 609 - 613]. For each $N \in \mathbb{N}$ and $\pi \in \Sigma_N$, define the unitary operator $U_\pi^{[N]} \in \mathcal{B}(\mathbb{H}^{\otimes N})$ by equation (4). For $A \in \mathcal{B}(\mathbb{H})$, $V \in \mathcal{B}(\mathbb{H} \otimes \mathbb{H})$, $N \in \mathbb{N}$, and $j \in \{1, ..., N\}$, define

$$A_j^{[N]} := 1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j-1)} \in \mathcal{B}(\mathbb{H}^{\otimes N}),$$

$$V_{12}^{[N]} := V \otimes 1^{\otimes(N-2)},$$

and

$$V_{ij}^{[N]} = U_{\pi^{-1}}^{[N]} V_{12}^{[N]} U_\pi^{[N]}$$

where π is any permutation where $\pi(i) = 1$ and $\pi(j) = 2$. Notice that this operator is well defined and independent of the permutation π that we use, (as long as $\pi(i) = 1$ and $\pi(j) = 2$) because when applied to a simple tensor $x_1 \otimes \cdots \otimes x_N$ all but the x_i and x_j spots are left invariant. For any self-adjoint $A \in \mathcal{B}(\mathbb{H})$, any self-adjoint $V \in \mathcal{B}(\mathbb{H} \otimes \mathbb{H})$, and each $N \in \mathbb{N}$, consider the Hamiltonian

$$H_N = \sum_{j=1}^{N} A_j^{[N]} + \frac{1}{N} \sum_{i \neq j; i,j=1}^{N} V_{ij}^{[N]}. \tag{16}$$

Also, define

$$H_{n,N} := \sum_{j=1}^{n} A_j^{[n]} + \frac{1}{N} \sum_{i \neq j; i,j=1}^{n} V_{ij}^{[n]} \tag{17}$$

for each $n, N \in \mathbb{N}$, $n \leq N$.

The main result of this section is Theorem 3.5. In this theorem, we will assume that a sequence of density operators $(\rho_N(0))_{N=1}^\infty$ is $\rho(0)$-chaotic and we will show that if $(H_N)_{N=1}^\infty$ is defined by equation (16) and for all $t \geq 0$, $(\rho_N(t))_{N=1}^\infty$ is defined by equation (15), then for all $t \geq 0$ the sequence $(\rho_N(t))_{N=1}^\infty$ is $\rho(t)$-chaotic for some $\rho(t) \in \mathcal{D}(\mathbb{H})$, i.e. chaos propagates with respect to $(H_N)_{N=1}^\infty$. Before proving our main result (Theorem 3.5), we need to establish some preliminary results. The first preliminary result consists of proving that $\rho_N(t)$ is symmetric for each $N \in \mathbb{N}$ and $t \geq 0$.

Proposition 3.2. *For each $N \in \mathbb{N}$ and $t \geq 0$, $\rho_N(t)$ (as defined in equation (15)) is symmetric.*

Proof. Let $\pi \in \Sigma_N$. By Proposition 2.3, we must show that $U_{\pi^{-1}}^{[N]} e^{-itH_N} \rho_N(0) e^{itH_N} U_\pi^{[N]} = U_{\pi^{-1}}^{[N]} \rho_N(t) U_\pi^{[N]} = \rho_N(t)$. Since $\rho_N(0)$ is symmetric, it is enough to show that $U_{\pi^{-1}}^{[N]} e^{itH_N} U_\pi^{[N]} = e^{itH_N}$. Furthermore, it is enough to show that $U_{\pi^{-1}}^{[N]} H_N U_\pi^{[N]} = H_N$.

First we prove that $U_{\pi^{-1}}^{[N]} \sum_{j=1}^{N} A_j^{[N]} U_{\pi}^{[N]} = \sum_{j=1}^{N} A_j^{[N]}$. Indeed,

$$\begin{aligned}
U_{\pi^{-1}}^{[N]} \sum_{j=1}^{N} A_j^{[N]} U_{\pi}^{[N]} (x_1 \otimes \cdots \otimes x_N) &= \sum_{j=1}^{N} U_{\pi^{-1}}^{[N]} A_j^{[N]} (x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(N)}) \\
&= \sum_{j=1}^{N} U_{\pi^{-1}}^{[N]} (x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(j-1)} \otimes A(x_{\pi^{-1}(j)}) \otimes x_{\pi^{-1}(j+1)} \otimes \cdots \otimes x_{\pi^{-1}(N)}) \\
&= \sum_{j=1}^{N} x_1 \otimes \cdots \otimes x_{\pi^{-1}(j)-1} \otimes A(x_{\pi^{-1}(j)}) \otimes x_{\pi^{-1}(j)+1} \otimes \cdots \otimes x_N \\
&= \sum_{j=1}^{N} x_1 \otimes \cdots \otimes x_{j-1} \otimes A x_j \otimes x_{j+1} \otimes \cdots \otimes x_N = \sum_{j=1}^{N} A_j^{[N]} (x_1 \otimes \cdots \otimes x_N).
\end{aligned}$$

Next, will will show that $U_{\pi^{-1}}^{[N]} \sum_{i \neq j; i,j=1}^{N} V_{ij}^{[N]} U_{\pi}^{[N]} = \sum_{i \neq j; i,j=1}^{N} V_{ij}^{[N]}$. For each $i, j \in \{1, ..., N\}$ with $i \neq j$, choose $\sigma_{ij} \in \Sigma_N$ with $\sigma_{ij}(i) = 1$ and $\sigma_{ij}(j) = 2$. Then

$$\begin{aligned}
U_{\pi^{-1}}^{[N]} \sum_{i \neq j; i,j=1}^{N} V_{ij}^{[N]} U_{\pi}^{[N]} &= \sum_{i \neq j; i,j=1}^{N} U_{\pi^{-1}}^{[N]} V_{ij}^{[N]} U_{\pi}^{[N]} = \sum_{i \neq j; i,j=1}^{N} U_{\pi^{-1}}^{[N]} U_{\sigma_{ij}^{-1}}^{[N]} V_{12}^{[N]} U_{\sigma_{ij}}^{[N]} U_{\pi}^{[N]} \\
&= \sum_{i \neq j; i,j=1}^{N} U_{(\sigma_{ij}\pi)^{-1}}^{[N]} V_{12}^{[N]} U_{\sigma_{ij}\pi}^{[N]}. \qquad (18)
\end{aligned}$$

Notice that $(\sigma_{ij}\pi)(\pi^{-1}(i)) = 1$ and $(\sigma_{ij}\pi)(\pi^{-1}(j)) = 2$, and thus, line (18) is equal to

$$\sum_{i \neq j; i,j=1}^{N} V_{\pi^{-1}(i)\pi^{-1}(j)}^{[N]} = \sum_{i \neq j; i,j=1}^{N} V_{ij}^{[N]},$$

where the last equality is valid because $i \neq j$ if and only if $\pi^{-1}(i) \neq \pi^{-1}(j)$. Thus, we obtain that $U_{\pi^{-1}}^{[N]} \sum_{i \neq j=1}^{N} V_{ij}^{[N]} U_{\pi}^{[N]} = \sum_{i \neq j=1}^{N} V_{ij}^{[N]}$. Hence, we have that $U_{\pi^{-1}}^{[N]} H_N U_{\pi}^{[N]} = H_N$, and $\rho_N(t)$ is symmetric. $\square$

We are aiming to construct two similar families of differential equations for $\left(\rho_N^{(n)}(t)\right)_{n=1}^{N-1}$ and $(\rho(t)^{\otimes n})_{n=1}^{\infty}$. The following Proposition gives a family of differential equations which is satisfied by $\left(\rho_N^{(n)}(t)\right)_{n=1}^{N-1}$.

Proposition 3.3. *Let $N \in \mathbb{N}$. For $n \in \mathbb{N}$, $n \leq N-1$, and $t \geq 0$, we have*

$$i\frac{d}{dt}\rho_N^{(n)}(t) = [H_{n,N}, \rho_N^{(n)}(t)] + \frac{N-n}{N}\sum_{j=1}^{n} tr_{\{n+1\}}[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, \rho_N^{(n+1)}(t)] \tag{19}$$

where $\rho_N(t)$ is given by (15) and $H_{n,N}$ is given by (17).

Proof. We know

$$i\frac{d}{dt}\rho_N(t) = [H_N, \rho_N(t)]\,.$$

Integrating both sides, we obtain

$$i\left(\rho_N(t) - \rho_N(0)\right) = \int_0^t [H_N, \rho_N(s)]\, ds.$$

Now, taking the partial trace of both sides, and using the fact that partial traces and integrals commute, we obtain

$$i\left(\rho_N^{(n)}(t) - \rho_N^{(n)}(0)\right) = \int_0^t \mathrm{tr}_{[n+1,N]}\left([H_N, \rho_N(s)]\right) ds. \tag{20}$$

We claim that

$$\begin{aligned} &\mathrm{tr}_{[n+1,N]}\left([H_N, \rho_N(s)]\right) \\ &= [H_{n,N}, \rho_N^{(n)}(s)] + \frac{N-n}{N}\sum_{j=1}^{n} \mathrm{tr}_{\{n+1\}}[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, \rho_N^{(n+1)}(s)] \end{aligned} \tag{21}$$

for each $s \in [0,\infty)$. In order to prove equation (21), fix $s \in [0,\infty)$, and by line [1, equation (2.11)], we need to prove that for every $B \in \mathcal{B}(\mathbb{H}^{\otimes n})$

$$\begin{aligned} \mathrm{tr}\left([H_N, \rho_N(s)]B \otimes 1^{\otimes(N-n)}\right) &= \mathrm{tr}\left(\left[\sum_{j=1}^{n} A_j^{[n]} + \frac{1}{N}\sum_{i\neq j=1}^{n} V_{ij}^{[n]}, \rho_N^{(n)}(s)\right] B\right) \\ &+ \frac{N-n}{N}\sum_{j=1}^{n} \mathrm{tr}\left(\mathrm{tr}_{\{n+1\}}[V_{j\,n+1}^{[n+1]} + V_{n+1,\,j}^{[n+1]}, \rho_N^{(n+1)}(s)]B\right). \end{aligned}$$

Let $B \in \mathcal{B}(\mathbb{H}^{\otimes n})$, and we have

$$\mathrm{tr}\left(\left[\sum_{j=1}^{N} A_j^{[N]} + \frac{1}{N}\sum_{i\neq j;i,j=1}^{N} V_{ij}^{[N]}, \rho_N(s)\right] B \otimes 1^{\otimes(N-n)}\right)$$

$$
\begin{aligned}
&= \operatorname{tr}\Bigg(\sum_{j=1}^{N} A_j^{[N]}\rho_N(s)\ B\otimes 1^{\otimes(N-n)} + \frac{1}{N}\sum_{i\neq j;i,j=1}^{N} V_{ij}^{[N]}\rho_N(s)\ B\otimes 1^{\otimes(N-n)} \\
&-\rho_N(s)\sum_{j=1}^{N} A_j^{[N]}\ B\otimes 1^{\otimes(N-n)} - \frac{1}{N}\rho_N(s)\sum_{i\neq j;i,j=1}^{N} V_{ij}^{[N]}\ B\otimes 1^{\otimes(N-n)}\Bigg) \\
&= \operatorname{tr}\Bigg(B\otimes 1^{\otimes(N-n)}\ \sum_{j=1}^{N} A_j^{[N]}\rho_N(s) + \frac{1}{N}B\otimes 1^{\otimes(N-n)}\ \sum_{i\neq j;i,j=1}^{N} V_{ij}^{[N]}\rho_N(s) \\
&-\rho_N(s)\sum_{j=1}^{N} A_j^{[N]}\ B\otimes 1^{\otimes(N-n)} - \frac{1}{N}\rho_N(s)\sum_{i\neq j;i,j=1}^{N} V_{ij}^{[N]}\ B\otimes 1^{\otimes(N-n)}\Bigg)
\end{aligned}
$$

$$
= \operatorname{tr}\left(B\otimes 1^{\otimes(N-n)}\ \sum_{j=1}^{N} A_j^{[N]}\rho_N(s) - \rho_N(s)\sum_{j=1}^{N} A_j^{[N]}\ B\otimes 1^{\otimes(N-n)}\right) \tag{22}
$$

$$
+\operatorname{tr}\left(\frac{1}{N}B\otimes 1^{\otimes(N-n)}\ \sum_{i\neq j;i,j=1}^{N} V_{ij}^{[N]}\rho_N(s) - \frac{1}{N}\rho_N(s)\sum_{i\neq j;i,j=1}^{N} V_{ij}^{[N]}\ B\otimes 1^{\otimes(N-n)}\right). \tag{23}
$$

Line (22) can be rewritten as

$$
\operatorname{tr}\left(B\otimes 1^{\otimes(N-n)}\ \sum_{j=1}^{n} A_j^{[N]}\rho_N(s)\right) - \operatorname{tr}\left(\rho_N(s)\sum_{j=1}^{n} A_j^{[N]}\ B\otimes 1^{\otimes(N-n)}\right) \tag{24}
$$

$$
+\operatorname{tr}\left(B\otimes 1^{\otimes(N-n)}\ \sum_{j=n+1}^{N} A_j^{[N]}\rho_N(s)\right) - \operatorname{tr}\left(\sum_{j=n+1}^{N} A_j^{[N]}\ B\otimes 1^{\otimes(N-n)}\rho_N(s)\right). \tag{25}
$$

Notice that $B\otimes 1^{\otimes(N-n)}\ \sum\limits_{j=n+1}^{N} A_j^{[N]} = \sum\limits_{j=n+1}^{N} A_j^{[N]}\ B\otimes I^{\otimes(N-n)}$, and so line (25) is equal to zero (even without taking the trace into account). Notice that in line (24), $A_j^{[N]} = A_j^{[n]}\otimes 1^{\otimes(N-n)}$

for $j \leq n$, thus line (24) can be written as

$$\begin{aligned}
&\operatorname{tr}\left(B\sum_{j=1}^{n} A_j^{[n]}\otimes 1^{\otimes(N-n)}\rho_N(s)\right)-\operatorname{tr}\left(\rho_N(s)\sum_{j=1}^{n} A_j^{[n]}B\ \otimes 1^{\otimes(N-n)}\right)\\
&=\operatorname{tr}\left(\rho_N^{(n)}(s)B\sum_{j=1}^{n} A_j^{[n]}\right)-\operatorname{tr}\left(\rho_N^{(n)}(s)\sum_{j=1}^{n} A_j^{[n]}B\right)\quad\text{(by [1, equation (2.11)])}\\
&=\operatorname{tr}\left(\left[\sum_{j=1}^{n} A_j^{[n]},\rho_N^{(n)}(s)\right]B\right).
\end{aligned}$$

Line (23) can be rewritten as

$$\begin{aligned}
&\operatorname{tr}\left(\frac{1}{N}B\otimes 1^{\otimes(N-n)}\sum_{i\neq j;i,j=1}^{n} V_{ij}^{[N]}\rho_N(s)\right)+\operatorname{tr}\left(\frac{1}{N}B\otimes 1^{\otimes(N-n)}\sum_{1\leq i\leq n<j\leq N} V_{ij}^{[N]}\rho_N(s)\right)\\
&+\operatorname{tr}\left(\frac{1}{N}B\otimes 1^{\otimes(N-n)}\sum_{1\leq j\leq n<i\leq N} V_{ij}^{[N]}\rho_N(s)\right)+\operatorname{tr}\left(\frac{1}{N}B\otimes 1^{\otimes(N-n)}\sum_{i\neq j;i,j=n+1}^{N} V_{ij}^{[N]}\rho_N(s)\right)\\
&-\operatorname{tr}\left(\frac{1}{N}\rho_N(s)\sum_{i\neq j;i,j=1}^{n} V_{ij}^{[N]}\ B\otimes 1^{\otimes(N-n)}\right)-\operatorname{tr}\left(\frac{1}{N}\rho_N(s)\sum_{1\leq i\leq n<j\leq N} V_{ij}^{[N]}\ B\otimes 1^{\otimes(N-n)}\right)\\
&-\operatorname{tr}\left(\frac{1}{N}\rho_N(s)\sum_{1\leq j\leq n<i\leq N} V_{ij}^{[N]}\ B\otimes 1^{\otimes(N-n)}\right)-\operatorname{tr}\left(\frac{1}{N}\rho_N(s)\sum_{i\neq j;i,j=n+1}^{N} V_{ij}^{[N]}\ B\otimes 1^{\otimes(N-n)}\right).
\end{aligned}$$

The first and fifth terms of the above expression give

$$\operatorname{tr}\left(\frac{1}{N}B\otimes 1^{\otimes(N-n)}\sum_{i\neq j;i,j=1}^{n} V_{ij}^{[N]}\rho_N(s)\right)-\operatorname{tr}\left(\frac{1}{N}\rho_N(s)\sum_{i\neq j;i,j=1}^{n} V_{ij}^{[N]}\ B\otimes 1^{\otimes(N-n)}\right). \tag{26}$$

The second and sixth terms of the same expression give

$$\operatorname{tr}\left(\frac{1}{N}B\otimes 1^{\otimes(N-n)}\sum_{1\leq i\leq n<j\leq N} V_{ij}^{[N]}\rho_N(s)\right)-\operatorname{tr}\left(\frac{1}{N}\rho_N(s)\sum_{1\leq i\leq n<j\leq N} V_{ij}^{[N]}B\otimes 1^{\otimes(N-n)}\right). \tag{27}$$

The third and seventh terms of the same expression give

$$\operatorname{tr}\left(\frac{1}{N}B\otimes 1^{\otimes(N-n)}\sum_{1\leq j\leq n<i\leq N} V_{ij}^{[N]}\rho_N(s)\right)-\operatorname{tr}\left(\frac{1}{N}\rho_N(s)\sum_{1\leq j\leq n<i\leq N} V_{ij}^{[N]}B\otimes 1^{\otimes(N-n)}\right). \tag{28}$$

The fourth and eighth terms of the same expression give

$$\text{(29)}\quad \operatorname{tr}\left(\frac{1}{N} B\otimes 1^{\otimes(N-n)} \sum_{i\neq j; i,j=n+1}^{N} V_{ij}^{[N]}\rho_N(s)\right) - \operatorname{tr}\left(\frac{1}{N}\sum_{i\neq j; i,j=n+1}^{N} V_{ij}^{[N]} B\otimes 1^{\otimes(N-n)}\rho_N(s)\right).$$

Notice that $B\otimes I^{\otimes(N-n)} \sum\limits_{i\neq j; i,j=n+1}^{N} V_{ij}^{[N]} = \sum\limits_{i\neq j; i,j=n+1}^{N} V_{ij}^{[N]}\ B\otimes I^{\otimes(N-n)}$, and so line (29) is equal to zero (even without taking the trace into account).

Notice that $V_{ij}^{[N]} = V_{ij}^{[n]}\otimes 1^{\otimes(N-n)}$ for $i,j\leq n,$, thus line (26) can be rewritten as

$$\begin{aligned}
&\frac{1}{N}\operatorname{tr}\left(B\sum_{i\neq j; i,j=1}^{n} V_{ij}^{[n]}\otimes 1^{\otimes(N-n)}\ \rho_N(s)\right) - \frac{1}{N}\operatorname{tr}\left(\sum_{i\neq j; i,j=1}^{n} V_{ij}^{[n]} B\otimes 1^{\otimes(N-n)}\ \rho_N(s)\right)\\
&= \frac{1}{N}\operatorname{tr}\left(\rho_N^{(n)}(s) B\sum_{i\neq j; i,j=1}^{n} V_{ij}^{[n]}\right) - \frac{1}{N}\operatorname{tr}\left(\rho_N^{(n)}(s)\sum_{i\neq j; i,j=1}^{n} V_{ij}^{[n]} B\right) \qquad \text{(by [1, equation (2.11)])}\\
&= \operatorname{tr}\left(\left[\frac{1}{N}\sum_{i\neq j=1}^{n} V_{ij}^{[n]}, \rho_N^{(n)}(s)\right] B\right).
\end{aligned}$$

There are $N-n$ values of j in line (27) and by symmetry of $\rho_N(s)$ we can replace all of these values of j by $n+1$ and thus we have that line (27) can be rewritten as

$$\frac{N-n}{N}\operatorname{tr}\left(B\otimes 1^{\otimes(N-n)}\ \sum_{i=1}^{n} V_{i\,n+1}^{[N]}\rho_N(s)\right) - \frac{N-n}{N}\operatorname{tr}\left(\sum_{i=1}^{n} V_{i\,n+1}^{[N]}\ B\otimes 1^{\otimes(N-n)}\ \rho_N(s)\right).$$

Notice that $V_{i\,n+1}^{[N]} = V_{i\,n+1}^{[n+1]}\otimes 1^{\otimes(N-(n+1))}$ for $i\leq n$, thus the last displayed equation is equal to

$$\begin{aligned}
&\frac{N-n}{N}\operatorname{tr}\left(\left(B\otimes 1\ \sum_{i=1}^{n} V_{i\,n+1}^{[n+1]}\right)\otimes 1^{\otimes(N-n-1)}\ \rho_N(s)\right)\\
&-\frac{N-n}{N}\operatorname{tr}\left(\left(\sum_{i=1}^{n} V_{i\,n+1}^{[n+1]}\ B\otimes 1\right)\ \otimes 1^{\otimes(N-n-1)}\ \rho_N(s)\right)
\end{aligned}$$

and therefore by [1, equation (2.11)] the last displayed expression is equal to

$$\begin{aligned}
&\frac{N-n}{N}\operatorname{tr}\left(B\otimes 1\ \sum_{i=1}^{n} V_{i\,n+1}^{[n+1]}\ \rho_N^{(n+1)}(s)\right) - \frac{N-n}{N}\operatorname{tr}\left(\sum_{i=1}^{n} V_{i\,n+1}^{(n+1)}\ B\otimes 1\ \rho_N^{(n+1)}(s)\right)\\
&= \frac{N-n}{N}\sum_{j=1}^{n}\operatorname{tr}\left([V_{j\,n+1}^{[n+1]}, \rho_N^{(n+1)}(s)]B\otimes 1\right) = \frac{N-n}{N}\sum_{j=1}^{n}\operatorname{tr}\left(\operatorname{tr}_{\{n+1\}}[V_{j\,n+1}^{[n+1]}, \rho_N^{(n+1)}(s)]B\right),
\end{aligned}$$

where again we used [1, equation (2.11)] to obtain the last equality.

Similarly, line (28) can be rewritten as

$$\frac{N-n}{N}\sum_{j=1}^{n}\mathrm{tr}\left(\mathrm{tr}_{\{n+1\}}[V^{[n+1]}_{n+1\,j},\rho_N^{(n+1)}(s)]B\right).$$

Thus, (20) and (21) lead to

$$i\left(\rho_N^{(n)}(t)-\rho_N^{(n)}(0)\right)=\int_0^t\left([H_{n,N},\rho_N^{(n)}(s)]+\frac{N-n}{N}\sum_{j=1}^{n}\mathrm{tr}_{\{n+1\}}[V^{[n+1]}_{j\,n+1}+V^{[n+1]}_{n+1\,j},\rho_N^{(n+1)}(s)]\right)ds.$$

We take the derivative of both sides to obtain the result

$$i\frac{d}{dt}\rho_N^{(n)}(t)=[H_{n,N},\rho_N^{(n)}(t)]+\frac{N-n}{N}\sum_{j=1}^{n}\mathrm{tr}_{\{n+1\}}[V^{[n+1]}_{j\,n+1}+V^{[n+1]}_{n+1\,j},\rho_N^{(n+1)}(t)].$$

□

The next proposition concludes with a family of differential equations which is satisfied by $(\rho(t)^{\otimes n})_{n=1}^{\infty}$. This family of differential equations is similar to the ones displayed in equation (19).

Proposition 3.4. *Let $t\geq 0$ and $\rho(0)\in\mathcal{D}(\mathbb{H})$. If $\rho(t)$ is the solution to the differential equation*

$$i\frac{d}{dt}\rho(t)=[A,\rho(t)]+tr_{\{2\}}\left[V^{[2]}_{12}+V^{[2]}_{21},\rho(t)\otimes\rho(t)\right]\tag{30}$$

(which is called the Hartree Equation), with initial condition $\rho(0)$, then we have that $(\rho(t)^{\otimes n})_{n=1}^{\infty}$ satisfies the family of differential equations

$$i\frac{d}{dt}\rho(t)^{\otimes n}=\sum_{j=1}^{n}\left[A^{[n]}_j,\rho(t)^{\otimes n}\right]+\sum_{j=1}^{n}tr_{\{n+1\}}\left[V^{[n+1]}_{j\,n+1}+V^{[n+1]}_{n+1\,j},\rho(t)^{\otimes(n+1)}\right].\tag{31}$$

Equation (30) has a unique solution, see [4, Theorem 4.1].

Proof. We have

$$\begin{aligned} i\frac{d}{dt}\rho(t)^{\otimes n}&=\sum_{j=1}^{n}\rho(t)^{\otimes(j-1)}\otimes i\frac{d}{dt}\rho(t)\otimes\rho(t)^{\otimes(n-j)}\quad(\text{“product” rule})\\ &=\sum_{j=1}^{n}\rho(t)^{\otimes(j-1)}\otimes\left([A,\rho(t)]+\mathrm{tr}_{\{2\}}\left[V^{[2]}_{12}+V^{[2]}_{21},\rho(t)\otimes\rho(t)\right]\right)\otimes\rho(t)^{\otimes(n-j)}\end{aligned}$$

by assumption. The last expression splits into the following two parts

$$\sum_{j=1}^{n} \rho(t)^{\otimes(j-1)} \otimes [A, \rho(t)] \otimes \rho(t)^{\otimes(n-j)} \tag{32}$$

$$+ \quad \sum_{j=1}^{n} \rho(t)^{\otimes(j-1)} \otimes \operatorname{tr}_{\{2\}} \left[V_{12}^{[2]} + V_{21}^{[2]}, \rho(t) \otimes \rho(t) \right] \otimes \rho(t)^{\otimes(n-j)} \tag{33}$$

Line (32) can be rewritten as

$$\begin{aligned} & [A, \rho(t)] \otimes \rho(t)^{\otimes(n-1)} + \rho(t) \otimes [A, \rho(t)] \otimes \rho(t)^{\otimes(n-1)} + \cdots + \rho(t)^{\otimes(n-1)} \otimes [A, \rho(t)] \\ = \; & \sum_{j=1}^{n} \left[A_j^{[n]}, \rho(t)^{\otimes n} \right]. \end{aligned} \tag{34}$$

We claim that for $j \leq n$,

$$\operatorname{tr}_{\{j+1\}} \left(\left(V_{j\,j+1}^{[n+1]} + V_{j+1\,j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) = \operatorname{tr}_{\{n+1\}} \left(\left(V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right). \tag{35}$$

Indeed, by [1, equation (2.11)], for any $B_1, ..., B_n \in \mathcal{B}(\mathbb{H})$, we have by the symmetry of $\rho(t)^{\otimes(n+1)}$,

$$\begin{aligned} & \operatorname{tr} \left(\operatorname{tr}_{\{j+1\}} \left(\left(V_{j\,j+1}^{[n+1]} + V_{j+1\,j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) B_1 \otimes \cdots \otimes B_n \right) \\ = \; & \operatorname{tr} \left(\left(V_{j\,j+1}^{[n+1]} + V_{j+1\,j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} B_1 \otimes \cdots \otimes B_j \otimes I \otimes B_{j+1} \otimes \cdots \otimes B_n \right) \\ = \; & \operatorname{tr} \left(B_1 \otimes \cdots \otimes B_j \otimes I \otimes B_{j+1} \otimes \cdots \otimes B_n \left(V_{j\,j+1}^{[n+1]} + V_{j+1\,j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) \\ = \; & \operatorname{tr} \left(B_1 \otimes \cdots \otimes B_n \otimes I \left(V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) \\ = \; & \operatorname{tr} \left(\operatorname{tr}_{\{n+1\}} \left(\left(V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) B_1 \otimes \cdots \otimes B_n \right) \end{aligned}$$

where for the second to last equality we used the symmetry of $\rho(t)^{\otimes(n+1)}$ to move each B_k (for $k \geq j+1$) to the kth spot and 1 to the $n+1$st spot.

Similar to equation (35), we have that for $j \leq n$,

$$\operatorname{tr}_{\{j+1\}} \left(\rho(t)^{\otimes(n+1)} \left(V_{j\,j+1}^{[n+1]} + V_{j+1\,j}^{[n+1]} \right) \right) = \operatorname{tr}_{\{n+1\}} \left(\rho(t)^{\otimes(n+1)} \left(V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]} \right) \right). \tag{36}$$

Line (33) can be rewritten as

$$
\begin{aligned}
& \sum_{j=1}^{n} \rho(t)^{\otimes(j-1)} \otimes \mathrm{tr}_{\{2\}} \left(\left(V_{12}^{[2]} + V_{21}^{[2]} \right) \rho(t) \otimes \rho(t) \right) \otimes \rho(t)^{\otimes(n-j)} \\
- & \sum_{j=1}^{n} \rho(t)^{\otimes(j-1)} \otimes \mathrm{tr}_{\{2\}} \left(\rho(t) \otimes \rho(t) \left(V_{12}^{[2]} + V_{21}^{[2]} \right) \right) \otimes \rho(t)^{\otimes(n-j)} \\
= & \sum_{j=1}^{n} \mathrm{tr}_{\{j+1\}} \left(\left(V_{j\,j+1}^{[n+1]} + V_{j+1\,j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} - \rho(t)^{\otimes(n+1)} \left(V_{j\,j+1}^{[n+1]} + V_{j+1\,j}^{[n+1]} \right) \right) \qquad (37)
\end{aligned}
$$

.

By equations (35) and (36), line (37) is equal to

$$
\begin{aligned}
& \sum_{j=1}^{n} \left(\mathrm{tr}_{\{n+1\}} \left(\left(V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) \right. \\
- & \left. \mathrm{tr}_{\{n+1\}} \left(\rho(t)^{\otimes(n+1)} \left(V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]} \right) \right) \right) \\
= & \sum_{j=1}^{n} \mathrm{tr}_{\{n+1\}} \left[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, \rho(t)^{\otimes(n+1)} \right]. \qquad (38)
\end{aligned}
$$

Of course (34) and (38) complete the proof. □

The similarity of the two equations (19) and (31) helps to prove the propagation of chaos presented in the following theorem. The idea of the proof of this theorem comes from Ducomet [6, Theorem 3.1], and Bardos, Golse, Gottlieb, and Mauser [2, Theorem 3.1].

Theorem 3.5. *Let a sequence $(\rho_N(0))_{N=1}^{\infty}$ of density operators be $\rho(0)$-chaotic where $\rho(0) \in \mathcal{D}(\mathbb{H})$. Let $(H_N)_{N=1}^{\infty}$ be a sequence of Hamiltonians defined by equation (16). Then, for each fixed $t \geq 0$, the sequence of density operators $(\rho_N(t))_{N=1}^{\infty}$ defined in equation (15) is $\rho(t)$-chaotic where $\rho(t)$ is the solution of the Hartree equation (equation (30)) with initial condition $\rho(0)$. Thus chaos propagates with respect to the Hamiltonians $(H_N)_{N=1}^{\infty}$.*

Proof. In order to prove Theorem 3.5, we will show the following: Fix $t_0 \geq 0$. Assume that $(\rho_N(t_0))_{N=1}^{\infty}$ is $\rho(t_0)$-chaotic where $\rho(t_0) \in \mathcal{D}(\mathbb{H})$. Then for $t \in \left[t_0, t_0 + \dfrac{1}{4||V||_\infty} \right)$, $(\rho_N(t))_{N=1}^{\infty}$ is $\rho(t)$-chaotic where $\rho(t) \in \mathcal{D}(\mathbb{H})$ is the solution to the Hartree equation (equation (30)) with initial condition $\rho(t_0)$.

For $t \in [t_0, \infty)$, notice that for each $n, N \in \mathbb{N}$ with $n \le N-1$, by Proposition 3.3

$$
\begin{aligned}
i\frac{d}{dt}\rho_N^{(n)}(t) &= [H_{n,N}, \rho_N^{(n)}(t)] + \frac{N-n}{N}\sum_{j=1}^{n} \mathrm{tr}_{\{n+1\}}[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, \rho_N^{(n+1)}(t)] \\
&= \sum_{j=1}^{n}\left[A_j^{[n]}, \rho_N^{(n)}(t)\right] + \sum_{j=1}^{n} \mathrm{tr}_{\{n+1\}}[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, \rho_N^{(n+1)}(t)] \\
&+ \frac{1}{N}\sum_{i\neq j=1}^{n}\left[V_{i\,j}^{[n]}, \rho_N^{(n)}(t)\right] - \frac{n}{N}\sum_{j=1}^{n} \mathrm{tr}_{\{n+1\}}[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, \rho_N^{(n+1)}(t)] \\
&= \mathcal{L}_n(\rho_N^{(n)}(t)) + \sum_{j=1}^{n} \mathrm{tr}_{\{n+1\}}[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, \rho_N^{(n+1)}(t)] + \epsilon_n(t, N, \rho_N(t_0)) \qquad (39)
\end{aligned}
$$

where $\mathcal{L}_n(\cdot) := \sum_{j=1}^{n}\left[A_j^{[n]}, \cdot\right]$ and

$$
\epsilon_n(t, N, \rho_N(t_0)) := \frac{1}{N}\sum_{i\neq j=1}^{n}\left[V_{i\,j}^{[n]}, \rho_N^{(n)}(t)\right] - \frac{n}{N}\sum_{j=1}^{n} \mathrm{tr}_{\{n+1\}}[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, \rho_N^{(n+1)}(t)].
$$

Also, by Proposition 3.4, for each $n \in \mathbb{N}$,

$$
i\frac{d}{dt}\rho(t)^{\otimes n} = \sum_{j=1}^{n}\left[A_j^{[n]}, \rho(t)^{\otimes n}\right] + \sum_{j=1}^{n} \mathrm{tr}_{\{n+1\}}\left[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, \rho(t)^{\otimes(n+1)}\right]. \qquad (40)
$$

Define $E_{n,N}(t) := \rho_N^{(n)}(t) - \rho(t)^{\otimes n}$ for each $n \le N$. Then, by subtracting (40) from (39), we obtain that for each $n, N \in \mathbb{N}$ with $n \le N-1$,

$$
i\frac{d}{dt}E_{n,N}(t) = \sum_{j=1}^{n}\left[A_j^{[n]}, E_{n,N}(t)\right] + \sum_{j=1}^{n} \mathrm{tr}_{\{n+1\}}\left[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, E_{n+1,N}(t)\right] + \epsilon_n(t, N, \rho_N(t_0)).
$$

Now, define $\mathcal{U}_{n,t}(\cdot) := e^{it\mathcal{L}_n}(\cdot) = e^{it\sum_{j=1}^{n} A_j^{[n]}}(\cdot)e^{-it\sum_{j=1}^{n} A_j^{[n]}}$. We claim that $\mathcal{U}_{n,t}$ is an isometry on the trace class operators on $\mathbb{H}^{\otimes n}$ for each $n \in \mathbb{N}$ and $t \in [0, \infty)$. Indeed, if $T \in \mathcal{B}(\mathbb{H}^{\otimes n})$ is a trace class operator, then

$$
||\mathcal{U}_{n,t}(T)||_1 = ||e^{it\sum_{j=1}^{n} A_j^{[n]}} T e^{-it\sum_{j=1}^{n} A_j^{[n]}}||_1 \le ||e^{it\sum_{j=1}^{n} A_j^{[n]}}||_\infty ||T||_1 ||e^{-it\sum_{j=1}^{n} A_j^{[n]}}||_\infty = ||T||_1,
$$

and similarly, by observing that $T = e^{-it\sum_{j=1}^{n} A_j^{[n]}}\mathcal{U}_{n,t}(T)e^{it\sum_{j=1}^{n} A_j^{[n]}}$, we get the reverse inequality.

Now define $Z_{n,N}(t) := \mathcal{U}_{n,t}(E_{n,N}(t))$ for $t \in \left[t_0, t_0 + \frac{1}{4||V||_\infty}\right)$. Then

$$\begin{aligned}
\frac{d}{dt} Z_{n,N}(t) &= i \sum_{j=1}^{n} A_j^{[n]} \mathcal{U}_{n,t}(E_{n,N}(t)) - i\mathcal{U}_{n,t}(E_{n,N}(t)) \sum_{j=1}^{n} A_j^{[n]} \\
&- i\mathcal{U}_{n,t}\left(\mathcal{L}_n(E_{n,N}(t)) + \sum_{j=1}^{n} \mathrm{tr}_{\{n+1\}} \left[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, E_{n+1,N}(t) \right] + \epsilon_n(t, N, \rho_N(t_0)) \right) \\
&= -i \sum_{j=1}^{n} \mathcal{U}_{n,t} \left(\mathrm{tr}_{\{n+1\}} \left[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, E_{n+1,N}(t) \right] \right) - i\mathcal{U}_{n,t}\left(\epsilon_n(t, N, \rho_N(t_0)) \right) \qquad (41)
\end{aligned}$$

where the last equality follows because $\mathcal{U}_{n,t}$ and $\mathcal{L}_n$ commute hence

$$i \sum_{j=1}^{n} A_j^{[n]} \mathcal{U}_{n,t}(E_{n,N}(t)) - i\mathcal{U}_{n,t}(E_{n,N}(t)) \sum_{j=1}^{n} A_j^{[n]} - i\mathcal{U}_{n,t}(\mathcal{L}_n(E_{n,N}(t))) = 0.$$

By integrating both sides of equation (41), we obtain that, for each $n, N \in \mathbb{N}$ with $n \leq N - 1$,

$$\begin{aligned}
Z_{n,N}(t) &= Z_{n,N}(t_0) - i \sum_{j=1}^{n} \int_{t_0}^{t} \mathcal{U}_{n,s} \left(\mathrm{tr}_{\{n+1\}} \left[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, E_{n+1,N}(s) \right] \right) ds \\
&- i \int_{t_0}^{t} \mathcal{U}_{n,s}\left(\epsilon_n(s, N, \rho_N(t_0)) \right) ds.
\end{aligned}$$

We will aim to show that $\lim_{N\to\infty} ||E_{n,N}(t)||_1 = 0$. We have

$$\begin{aligned}
& ||E_{n,N}(t)||_1 = ||Z_{n,N}(t)||_1 \leq ||Z_{n,N}(t_0)||_1 \\
+ & \left|\left| \sum_{j=1}^{n} \int_{t_0}^{t} \mathcal{U}_{n,s} \left(\mathrm{tr}_{\{n+1\}} \left[V_{j\,n+1}^{[n+1]} + V_{n+1\,j}^{[n+1]}, E_{n+1,N}(s) \right] \right) ds \right|\right|_1 \\
+ & \left|\left| \int_{t_0}^{t} \mathcal{U}_{n,s}\left(\epsilon_n(s, N, \rho_N(t_0)) \right) \right|\right|_1 \\
\leq & \; ||E_{n,N}(t_0)||_1 + (t - t_0)||\epsilon_n(s, N, \rho_N(t_0))||_1 + 4||V||_\infty \sum_{j=1}^{n} \int_{t_0}^{t} ||E_{n+1,N}(s)||_1 \, ds. \qquad (42)
\end{aligned}$$

We notice that for every $n, N \in \mathbb{N}$ with $n \leq N-1$ and $s \in [0,\infty)$,

$$\begin{aligned}
& ||\epsilon_n(s,N,\rho_N(t_0))||_1 \\
= & \left|\left| \frac{1}{N}\sum_{i\neq j=1}^{n}\left[V_{i\,j}^{[n]},\rho_N^{(n)}(s)\right] - \frac{n}{N}\sum_{j=1}^{n}\mathrm{tr}_{\{n+1\}}[V_{j\,n+1}^{[n+1]}+V_{n+1\,j}^{[n+1]},\rho_N^{(n+1)}(s)]\right|\right|_1 \\
\leq & \frac{1}{N}\left|\left|\sum_{i\neq j;i,j=1}^{n}\left[V_{i\,j}^{[n]},\rho_N^{(n)}(s)\right]\right|\right|_1 + \frac{n}{N}\left|\left|\sum_{j=1}^{n}\mathrm{tr}_{\{n+1\}}[V_{j\,n+1}^{[n+1]}+V_{n+1\,j}^{[n+1]},\rho_N^{(n+1)}(s)]\right|\right|_1 \\
\leq & \frac{n(n-1)}{N}||V||_\infty + \frac{4n^2}{N}||V||_\infty \leq \frac{5n^2}{N}||V||_\infty. \qquad (43)
\end{aligned}$$

Inequalities (42) and (43) give

$$||E_{n,N}(t)||_1 \leq ||E_{n,N}(t_0)||_1 + \frac{5n^2}{N}||V||_\infty(t-t_0) + 4||V||_\infty\sum_{j=1}^{n}\int_{t_0}^{t}||E_{n+1,N}(s)||_1 ds.$$

Fixing $n \in \mathbb{N}$ and iterating this inequality m more times for $m \in \mathbb{N}$ and $m \leq N-n-1$, we obtain

$$\begin{aligned}
(44)\, & ||E_{n,N}(t)||_1 \leq ||E_{n,N}(t_0)||_1 + \frac{5n^2}{N}||V||_\infty(t-t_0) \\
+ & \sum_{k=1}^{m}(4||V||_\infty)^k\left[\sum_{j_1=1}^{n}\sum_{j_2=1}^{n+1}\cdots\sum_{j_k=1}^{n+k-1}\left(\frac{(t-t_0)^k}{k!}||E_{n+k,N}(0)||_1 + \frac{5(n+k)^2}{N}||V||_\infty\frac{(t-t_0)^{k+1}}{(k+1)!}\right)\right] \\
+ & (4||V||_\infty)^{m+1}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n+1}\cdots\sum_{j_{m+1}=1}^{n+m}\int_{t_0}^{t}\int_{t_0}^{t_1}\cdots\int_{t_0}^{t_m}||E_{n+m+1,N}(t_{m+1})||_1\, dt_{m+1}dt_m\cdots dt_1.
\end{aligned}$$

Since, for $i \leq j$, $E_{i,j}(t)$ is equal to a difference of two density operators, its trace class norm is less than or equal to two. Thus the last line of inequality (44) can be bounded above by

$$\begin{aligned}
& (4||V||_\infty)^{m+1}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n+1}\cdots\sum_{j_{m+1}=1}^{n+m}\int_{t_0}^{t}\int_{t_0}^{t_1}\cdots\int_{t_0}^{t_m}2dt_{m+1}dt_m\cdots dt_1 \\
= & 2(4||V||_\infty)^{m+1}n(n+1)\cdots(n+m)\frac{(t-t_0)^{m+1}}{(m+1)!} \\
= & 2\frac{n(n+1)\cdots(n+m)}{(m+1)!}(4||V||_\infty(t-t_0))^{m+1} = 2\binom{n+m}{n-1}(4||V||_\infty(t-t_0))^{m+1} \\
\leq & \frac{2}{n!}(n+m)^n(4||V||_\infty(t-t_0))^{m+1}
\end{aligned}$$

where we used that

$$\binom{n+m}{n-1} = \frac{(n+m)!}{(n-1)!(m+1)!} \leq \frac{(n+m)^{n-1}}{(n-1)!} = \frac{n(n+m)^{n-1}}{n!} \leq \frac{(n+m)^n}{n!}.$$

Thus, by (44), we obtain

$$\begin{aligned} & ||E_{n,N}(t)||_1 \leq ||E_{n,N}(t_0)||_1 + \frac{5n^2}{N}||V||_\infty (t-t_0) \\ + & \sum_{k=1}^{m} (4||V||_\infty)^k \left[\sum_{j_1=1}^{n} \sum_{j_2=1}^{n+1} \cdots \sum_{j_k=1}^{n+k-1} \left(\frac{(t-t_0)^k}{k!} ||E_{n+k,N}(t_0)||_1 + \frac{5(n+k)^2}{N} ||V||_\infty \frac{(t-t_0)^{k+1}}{(k+1)!} \right) \right] \\ + & \frac{2}{n!}(n+m)^n (4||V||_\infty (t-t_0))^{m+1}. \end{aligned}$$

Let $\epsilon > 0$. Fix $t \in \left[t_0, t_0 + \frac{1}{4||V||_\infty}\right)$. Choose m such that

$$\frac{2}{n!}(n+m)^n (4||V||_\infty (t-t_0))^{m+1} < \frac{\epsilon}{3}.$$

Then since $\lim_{N\to\infty} ||E_{n,N}(t_0)||_1 = 0$ by Proposition 2.8, we can choose $N_1 \in \mathbb{N}$ large enough such that

$$||E_{n,N}(t_0)||_1 + \sum_{k=1}^{m} (4||V||_\infty)^k \sum_{j_1=1}^{n} \sum_{j_2=1}^{n+1} \cdots \sum_{j_k=1}^{n+k-1} \frac{(t-t_0)^k}{k!} ||E_{n+k,N}(t_0)||_1 < \frac{\epsilon}{3}$$

for all $N \geq N_1$.

Then choose $N_2 \in \mathbb{N}$ such that

$$\frac{5n^2}{N}||V||_\infty (t-t_0) + \sum_{k=1}^{m} (4||V||_\infty)^k \sum_{j_1=1}^{n} \sum_{j_2=1}^{n+1} \cdots \sum_{j_k=1}^{n+k-1} \frac{5(n+k)^2}{N} ||V||_\infty \frac{(t-t_0)^{k+1}}{(k+1)!} < \frac{\epsilon}{3}$$

for all $N \geq N_2$. For $N \geq \max\{N_1, N_2\}$,

$$||E_{n,N}(t)||_1 < \epsilon,$$

i.e. $\lim_{N\to\infty} ||E_{n,N}(t)||_1 = 0$, and $\rho_N^{(n)}(t)$ is $\rho(t)$-chaotic for all $t \in \left[t_0, t_0 + \frac{1}{4||V||_\infty}\right)$.

□

References

[1] S. Attal, *Tensor Products and Partial Traces*, Lecture 2.
[2] C. Bardos, F. Golse, A. Gottlieb, and N. Mauser, *Mean field dynamics of fermions and the time-dependent hartree-fock equation*, Mathematics Pures Applications, 82, (2003), 665-683.
[3] L. Boltzmann, J. Math. Vol. 100, (1887) pp. 201.
[4] A. Bove, G. Da Prato, G. Fano, *An existence proof for the Hartree-Fock time-dependent problem with bounded two-body interaction*, Communications in Mathematical Physics 37: 183-191, 1974.
[5] E. Carlen, M.C. Carvalho, M. Loss, *Kinetic theory and the Kac master equation*, Entropy & Quantum II, Contemp. Math. **552**, (2011), 1-20.
[6] B. Ducomet, *Weak interaction limit for nuclear matter and the time-dependent hartree-fock equation*, Applications of Mathematics, Vol. 55 No. 3, (2010), 197-219.
[7] F. Golse, T. Paul, *Empirical measures and quantum mechanics: application to the mean-field limit*, arXiv:1711.08350v1 [math.AP], 22 November 2017.
[8] A. Gottlieb, *Classical and Quantum Propagation of Chaos*, arXiv:quant-ph/0108143v1, August 2001.
[9] G. Graham, S. Méléard, *Stochastic particle approximations for generalized Boltzmann models and convergence estimates*, Ann. Propab. **25** (1997), 115-132.
[10] F.A. Grünbaum, *Propagation of chaos for the Boltzmann equation*, Arch. Rational Mech. Anal. **42** (1971), 323-345.
[11] D.P. Johnson, *On a class of stochastic processes and its relationship to infinite particle gases*, Trans. Amer. Math. Soc. **132** (1968), 275-295.
[12] M. Kac, *Foundations of Kinetic Theory*, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability: held at the Statistical Laboratory, University of California, (1956).
[13] H.P. McKean, *Propagation of chaos for a class of non-linear parabolic equations*, Stochastic Differential Equations, (Lecture Series in Differential Equations Session 7, Catholic University, 1967), 41-57, Air Force Office Sci. Res. Arlington, Va.
[14] S. Mischler, *Le programme de Kac sur les limites de champ moyen*, in: Séminaire EDP-X, Décembre 2010.
[15] S. Mischler, C. Mouhot, *Kac's program in kinetic theory*, Invent. Math. **193** (1) (2014) 1-147.
[16] H. Spohn, *Kinetic equations from Hamiltonian dynamics: Markovian limits*, Reviews of Modern Physics, Vol. 53, No. 3, (1980), pp. 569-615.
[17] A. Sznitman, *Equations de type de Boltzmann, spatialement homogenes*, Laboratoire de Probabilites 224 (1984), pp. 559-592.
[18] A. Sznitman, *Topics in Propagation of Chaos*, Lecture Notes in Mathematics 1464 (1991), pp. 165-251.
[19] H. Tanaka, *Propagation of chaos for certain Markov processes of jump type with non-linear generators*, Part I, Part II, Proc. Japan Acad. **45** (1969), 449-452, 598-599.
[20] T. Ueno, *A class of Markov processes with non-linear bounded generators*, Japan J. Math. **38**, (1969), 19-38.
[21] A. Wehrl, *Three Theorems about entropy and convergence of density matrices*, Reports on Mathematical Physics, Vol. 10, No. 2, (October 1976), pp. 159-163.

Department of Mathematics, University of South Carolina, Columbia, SC 29208
E-mail address: giorgis@math.sc.edu

Department of Mathematics, University of South Carolina, Columbia, SC 29208
E-mail address: musulin@math.sc.edu